Lecture 5

Often of importance is the intersection of surfaces. The result is generically a curve (e.g., the intersection of two planes is a line, as we know). It is important to be able to quantify these "curves of intersection" by <u>parametrizing</u> them (finding a vector-valued function whose image is the curve).

<u>Ex</u>: Find a vector function representing the intersection of $z=x^2$ and $x^2+y^2=4$.

<u>Sol</u>: We know that whatever the vector function $\vec{r}(t)=\langle x(t), y(t), z(t)\rangle$ is, it must satisfy the equations of the surfaces since it lies on both of them, that is, we must have

$$z(t)=[x(t)]^2 \text{ and } [x(t)]^2+[y(t)]^2=4.$$

The second equation suggests taking

$$x(t)=2\cos t \;\&\; y(t)=2\sin t$$

and the first then gives $z(t)=4\cos^2 t$.

Thus the vector equation is

$$\vec{r}(t)=\langle 2\cos t, 2\sin t, 4\cos^2 t\rangle$$

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13.2 - Derivatives and Integrals

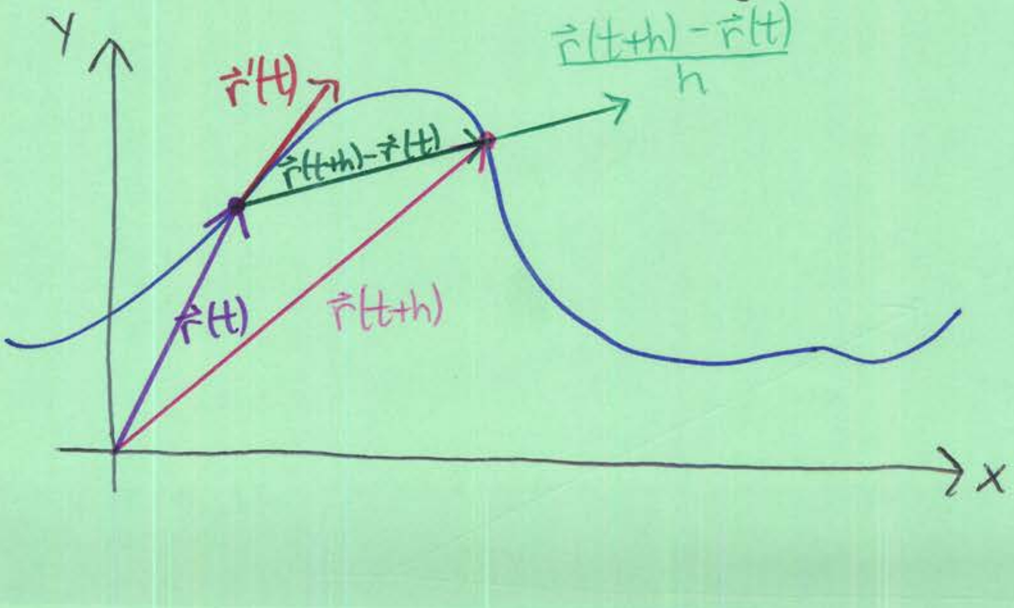
What were the main applications of limits back in Calc I?

Differentiation and integration!

Def: The derivative of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is:

$$\boxed{\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle}$$

Recall that the derivative of a function gave us the slope of its tangent line. For vector-valued functions, the derivative gives us a tangent vector (pointing in the direction of increasing t-values). This vector can be used as a direction vector for the tangent line.



$\vec{r}'(a)$ is thus the <u>tangent vector to $\vec{r}(t)$ at $t=a$</u> (provided $\vec{r}'(a) \neq \vec{0}$). A very important variation of the tangent vector is the <u>unit tangent vector</u>:

$$\boxed{\vec{T}(t) := \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}}$$

This will be important in the next section.

<u>Ex</u>: Let $\vec{r}(t) = \langle t\cos t, t, t\sin t\rangle$. Find:
i) $\vec{r}'(t)$, ii) $\vec{T}(t)$, iii) an equation for the tangent line to $\vec{r}(t)$ at $t=\pi$.

<u>Sol</u>: i) $\vec{r}'(t) = \langle \cos t - t\sin t, 1, \sin t + t\cos t\rangle$

ii) First, we need $\|\vec{r}'(t)\|$:

$$\|\vec{r}'(t)\| = \sqrt{(\cos t - t\sin t)^2 + 1 + (\sin t + t\cos t)^2}$$

$$= \sqrt{\cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + 1 + \sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t}$$

$$= \sqrt{1 + t^2 + 1} = \sqrt{t^2+2}$$

So, $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \dfrac{1}{\sqrt{t^2+2}}\langle \cos t - t\sin t, 1, \sin t + t\cos t\rangle$

iii) A direction vector for the tangent line at $t=\pi$ is

$$\vec{r}\,'(\pi) = \langle \cos\pi - \pi\sin\pi, 1, \sin\pi + \pi\cos\pi \rangle = \langle -1, 1, -\pi \rangle$$

(note that we could have also used $\vec{T}(\pi)$)

A point on the tangent line is $\vec{r}(\pi) = \langle -\pi, \pi, 0 \rangle$, so an equation for the tangent line is:

$$\vec{\ell}(s) = \vec{r}(\pi) + s\,\vec{r}\,'(\pi) = \langle -\pi - s, \pi + s, -\pi s \rangle$$

$\diamond$

Properties of Derivatives: $\vec{u}(t), \vec{v}(t)$: vector functions
c constant, $f(t)$ scalar function

1) $\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}\,'(t) + \vec{v}\,'(t)$
2) $\frac{d}{dt}\left[c\vec{u}(t)\right] = c\vec{u}\,'(t)$

} linearity of $\frac{d}{dt}$

3) $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$
4) $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$
5) $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

} Product rule

6) $\frac{d}{dt}\left[\vec{u}(f(t))\right] = \vec{u}\,'(f(t))f'(t)$ } Chain rule

There is a useful consequence of #4: Suppose $\vec{r}\,'(t) \neq \vec{0}$

$$\frac{d}{dt}\|\vec{r}(t)\| = \frac{d}{dt}\sqrt{\vec{r}(t)\cdot\vec{r}(t)} = \frac{1}{2}\,\frac{\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)]}{\sqrt{\vec{r}(t)\cdot\vec{r}(t)}}$$

$$= \frac{1}{2}\,\frac{2\vec{r}(t)\cdot\vec{r}\,'(t)}{\|\vec{r}(t)\|} = \frac{\vec{r}(t)\cdot\vec{r}\,'(t)}{\|\vec{r}(t)\|}$$

Finally, integration:

<u>Def</u>: The <u>definite integral</u> of $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is

$$\int_a^b \vec{r}(t)\,dt = \lim_{n\to\infty}\sum_{i=1}^{n}\vec{r}(t_i^*)\Delta t_i$$

$$= \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

The t_i^* are chosen from the i^{th} piece of a partition of $[a,b]$ into n pieces. One can also define <u>indefinite integrals</u>:

$$\int \vec{r}(t)\,dt = \left(\int f(t)\,dt\right)\hat{\imath} + \left(\int g(t)\,dt\right)\hat{\jmath} + \left(\int h(t)\,dt\right)\hat{k}$$

<u>Ex</u>: Find $\int \vec{r}(t)\,dt$ and $\int_0^2 \vec{r}(t)\,dt$ where $\vec{r}(t) = t\hat{\imath} - t^3\hat{k}$.

<u>Sol</u>: $\int \vec{r}(t)\,dt = \left(\int t\,dt\right)\hat{\imath} + \left(\int 0\,dt\right)\hat{\jmath} + \left(\int -t^3\,dt\right)\hat{k}$

$$= \left(\frac{1}{2}t^2 + c_1\right)\hat{\imath} + c_2\hat{\jmath} + \left(-\frac{1}{4}t^4 + c_3\right)\hat{k}$$

$$\int_0^2 \vec{r}(t)\,dt = \left(\frac{1}{2}t^2\Big|_0^2\right)\hat{\imath} + 0\hat{\jmath} + \left(-\frac{1}{4}t^4\Big|_0^2\right)\hat{k} = 2\hat{\imath} - 4\hat{k}$$

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